

BOUNDING THE NUMBER OF MAXIMAL TORSION COSETS ON SUBVARIETIES OF ALGEBRAIC TORI

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ABSTRACT. We obtain bounds on the number of maximal torsion cosets for algebraic subvarieties $V \subset \mathbb{G}_m^n$, defined over \mathbb{Q} , using model theoretic methods.

Let V be an algebraic variety defined over the rationals \mathbb{Q} , with $V \subset \mathbb{G}_m^n(\mathbb{C})$, where $\mathbb{G}_m^n(\mathbb{C})$ is the multiplicative subgroup of \mathbb{C}^n . A *cyclotomic point* $\bar{\omega}$ on V is a point of the form $(\omega_1, \dots, \omega_n)$, with ω_i a root of unity for $1 \leq i \leq n$. A *torsion coset* is a set of the form $\bar{\omega}T$ where T is a connected algebraic subgroup of $\mathbb{G}_m^n(\mathbb{C})$. Any connected algebraic subgroup T of $\mathbb{G}_m^n(\mathbb{C})$ is isomorphic to $\mathbb{G}_m^r(\mathbb{C})$ for some $0 \leq r \leq n$. We can write the isomorphism Φ in the following form;

$$\begin{aligned} \Phi : \mathbb{G}_m^r(\mathbb{C}) &\rightarrow T \\ (t_1, \dots, t_r) &\mapsto (t_1^{m_{11}} \dots t_r^{m_{1r}}, \dots, t_1^{m_{n1}} \dots t_r^{m_{nr}}) \end{aligned}$$

We let $\mathbf{M} = (m_{ij})_{1 \leq i \leq n, 1 \leq j \leq r}^1$ be the matrix defining the isomorphism Φ , so we can write an element of a torsion coset in the form $\bar{\omega} \mathbf{t}^{\mathbf{M}}$, where $\mathbf{t} = (t_1, \dots, t_r) \in \mathbb{G}_m^r(\mathbb{C})$. We define a torsion coset $\bar{\omega}T \subset V$ to be *maximal* if for any torsion coset $\bar{\omega}'T' \subset V$, with $\bar{\omega}T \subset \bar{\omega}'T'$, we have that $\bar{\omega}T = \bar{\omega}'T'$.

We make the following straightforward observations about maximal torsion cosets. First,

Lemma 0.1. $\overline{V(K^{cycl})} = \bigcup_{i=1}^N \bar{\omega}_i T_i$

where i runs over all the maximal torsion cosets on V .

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This is clear by noting that a cyclotomic point on V is itself a torsion coset, hence we obtain the left hand inclusion. We obtain equality by the fact that cyclotomic points are Zariski dense in any torsion coset. Secondly, if $\bar{a}S \subseteq V$ is a coset of a connected algebraic subgroup of \mathbb{G}_m^n , then;

$$\overline{\bar{a}S(K^{cycl})} = \bar{w}T$$

where T is a connected algebraic subgroup of $\mathbb{G}_m^n(K)$ and \bar{w} is a cyclotomic point.

In order to see this, observe that $\overline{\bar{a}S(K^{cycl})} = \bigcup_{i=1}^N \bar{w}_i T_i$, where i runs over all the maximal torsion cosets on $\bar{a}S$. We claim that in fact $N = 1$. Let \bar{w} be a cyclotomic point on $\bar{a}S$, then we have that $\bar{w}_1^{-1}\bar{w} \in S$ and $\bar{w} \in \bar{w}_1 S$. Hence $\bar{a}S(K^{cycl}) = (\bar{w}_1 S)(K^{cycl}) = \bar{w}_1(S(K^{cycl}))$. We now claim that $\overline{S(K^{cycl})}$ is connected, this follows easily from the fact that S is a connected algebraic subgroup of $\mathbb{G}_m^n(K)$ and the classification of such groups. It follows that $\overline{\bar{a}S(K^{cycl})} = \bar{w}_1 T'_1$, but, by maximality of $\bar{w}_1 T_1$, we have that $T_1 = T'_1$ as required.

An important consequence of the above result is the following;

Lemma 0.2. *Let $\bar{w}_i T_i$, for $1 \leq i \leq n$, enumerate the maximal torsion cosets on V and suppose that;*

$$\bigcup_{i=1}^N \bar{w}_i T_i \subseteq \bigcup_{i=1}^M \bar{a}_i S_i \subseteq V$$

where the S_i are connected algebraic subgroups of $\mathbb{G}_m^n(K)$. Then $N \leq M$.

Proof. We have that $\bigcup_{i=1}^N \bar{w}_i T_i = \overline{\bigcup_{i=1}^M \bar{a}_i S_i(K^{cycl})} = \bigcup_{i=1}^M \overline{\bar{a}_i S_i(K^{cycl})}$. By the previous argument, this is a union of M irreducible torsion cosets. As the irreducible decomposition of $\bigcup_{i=1}^N \bar{w}_i T_i$ is irredundant, by maximality of each torsion coset $\bar{w}_i T_i$, we have that $N \leq M$ as required.

□

The following result is given in [4], (Theorem 8E*);

Suppose that V is defined by equations of total degree $\leq d$, then there exist finitely many maximal torsion cosets on V and, moreover,

the number is bounded by $\exp(3N(d)^{\frac{3}{2}} \log(N(d)))$ where $N(d) = C_d^{n+d}$.

The purpose of this paper is to find an alternative bound in the degree d and dimension n , using methods from model theory.

We define the exponent of a torsion coset $\bar{\omega}T$ to be any multiple of its order as an element of the group \mathbb{G}_m^n/T . By the result given above, there must exist a single exponent for all maximal torsion cosets on V . In [3], (Theorem 4.2), the following result is proved;

Suppose that $N(V)$, the Newton polygon associated to V , has diameter $D(V)$. Then every $(n-k)$ -dimensional maximal torsion coset on V has an exponent mP_N for $m \leq D(V)^{2k}k^{k/2}$, where $N = \text{Card}(N(V)) \leq C_d^{m+d}$ and P_N is the product of all primes up to N . (By results of Iskander Aliev, the bound on m can be improved to $D(V)^{2k}$.)

Now let K be a uniform exponent for all the maximal torsion cosets on V , for example we can take $K = tP_N$, where $t \leq D(V)^{n(n-1)}$, assuming the result of Aliev. Suppose that we are given a maximal torsion coset $\bar{\omega}T$ of exponent K , then we have that $\bar{\omega}^K \in T$. Suppose that $\Phi(\bar{t}) = \bar{\omega}^K$. Clearly, $\mathbb{G}_m^r(\mathbb{C})$ is closed under taking K 'th roots, so we can find $\bar{t}_1 \in \mathbb{G}_m^r(\mathbb{C})$ such that $\bar{t}_1^K = \bar{t}$. Then $\Phi(\bar{t}_1^K) = \Phi(\bar{t}_1)^K = \bar{\omega}^K$. Therefore, $(\Phi(\bar{t}_1)^{-1}\bar{\omega})^K = \mathbf{1}$ and $\Phi(\bar{t}_1)^{-1}\bar{\omega}$ represents the same torsion coset as $\bar{\omega}$. Enumerate elements $\{\bar{w}_1, \dots, \bar{w}_N\}$ representing the maximal torsion cosets on V such that $\bar{w}_j^K = \mathbf{1}$ for $1 \leq j \leq N$. We can write each \bar{w}_j as $(\omega_{1j}, \dots, \omega_{nj})$ where ω_{ij} is a primitive L_{ij} 'th root of unity with $L_{ij} | K$, for $1 \leq i \leq n$. Now choose a primitive K 'th root of unity ξ with $\xi^{K/L_{ij}} = \omega_{ij}$. We consider the following cyclotomic extensions;

(i). $\mathbb{Q}(\xi)/\mathbb{Q}$;

In this case $\text{Gal}(\mathbb{Q}(\xi)/\mathbb{Q}) = U_{p_1^{m_1}} \times \dots \times U_{p_r^{m_r}}$, where $K = p_1^{m_1} \dots p_r^{m_r}$ is the prime factorisation of K and $U_{p_j^{m_j}}$ is the cyclic group of units in the multiplicative group $(\mathbb{Z}/p_j^{m_j}\mathbb{Z})^*$. This is an abelian group with generators $\{\sigma_1, \dots, \sigma_r\}$.

a. If K is odd, $(2, K) = 1$, hence $2 \in U_K$ and the map $\sigma(\xi) = \xi^2$ determines an element of $\text{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$. We then have that $\sigma(\omega_{ij}) = \omega_{ij}^2$ as well.

b. If $K = 2L$ with L odd, then $(L + 2, 2L) = 1$ and $\sigma(\xi) = \xi^{L+2} = -\xi^2$ determines an element of $\text{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$, then $\sigma(\omega_{ij}) = (-1)^{K/L_{ij}} \omega_{ij}^2$.

c. If $K = 4L$, then $(2L + 1, 4L) = 1$ and $\sigma(\xi) = \xi^{2L+1} = -\xi$ determines an element of $\text{Gal}(\mathbb{Q}(\xi)/\mathbb{Q})$, then $\sigma(\omega_{ij}) = (-1)^{K/L_{ij}} \omega_{ij}$.

We extend σ to a generic automorphism of $\mathbb{Q}(\xi) \subset K$, hence $(K, \sigma) \models \text{ACFA}$.

In the three cases, we construct a functional equation in σ . We denote the coordinates on \mathbb{G}_m^n by (x_1, \dots, x_n)

$$(a,b).((\sigma(x_1) - x_1^2)(\sigma(x_1) + x_1^2), \dots, (\sigma(x_n) - x_n^2)(\sigma(x_n) + x_n^2)) = \mathbf{0} \quad (1)$$

$$(c). ((\sigma(x_1) - x_1)(\sigma(x_1) + x_1), \dots, (\sigma(x_n) - x_n)(\sigma(x_n) + x_n)) = \mathbf{0} \quad (*)$$

If we denote the action by σ on $\mathbb{G}_m^n(K)$ as $\sigma(x_1, \dots, x_n) = (\sigma(x_1), \dots, \sigma(x_n))$, and for $m \in \mathbb{Z}$, define $m(x_1, \dots, x_n) = (x_1^m, \dots, x_n^m)$, then any polynomial $P(X)$ with integer coefficients defines an endomorphism of $\mathbb{G}_m^n(K)$. We can then write the functional equation

$$(\sigma(x_1) - x_1^p, \dots, \sigma(x_n) - x_n^p) = \mathbf{0}$$

as $\text{Ker}(P(\sigma))$ where $P(X)$ denotes the polynomial $X - p$.

We will denote the subgroups of $\mathbb{G}_m^n(K)$ defined by the polynomial $X - p$ for $p \in \mathbb{Z}$ by \mathbb{G}_p .

The functional equations from $(*)$ define subgroups of $\mathbb{G}_m^n(K)$, which we denote by \mathbb{G}^1 and \mathbb{G}^2 .

We need the following definitions;

Let A be a σ -definable subgroup of $\mathbb{G}_m^n(K)$. We say that A is LMS (stable, stably embedded and 1-based) if every σ -definable subset X of A^r (possibly with parameters outside A) is a finite Boolean combination of cosets of definable subgroups of A^r . We say that A is algebraically modular (ALM), if for every σ definable $X \subset A^r$, the Zariski closure of X is a finite union of cosets of algebraic subgroups of $\mathbb{G}_m^n(K)$

The following result is proved in [2] (Corollary 4.1.13);

Let $p(T)$ be a polynomial with integer coefficients defining an endomorphism $p(\sigma)$ of $\mathbb{G}_m^n(K)$. Then $\text{Ker}(p(\sigma))$ is LMS iff $p(T)$ has no cyclotomic factors.

Applying this result to the polynomial $X - p$ for $p \in \mathbb{Z}$ gives that the groups \mathbb{G}_p are LMS iff $p \notin \{1, -1\}$.

If A is LMS, then A is ALM. This is almost immediate from the definitions. Let $X \subset A^r$ be σ -definable. As A is LMS, we can write;

$$X = \bigcup_{i=1}^n (C_i \setminus D_i)$$

where the C_i are disjoint cosets of groups $H_i \subset A^r$ and D_i is contained in a finite union of cosets of subgroups of H_i of infinite index. We then clearly have that;

$$\overline{X} = \overline{\bigcup_{i=1}^n (C_i \setminus D_i)} = \bigcup_{i=1}^n \overline{C_i}$$

Each $\overline{C_i}$ is clearly an algebraic subgroup of $\mathbb{G}_m^n(K)$, which gives the result.

In particular, the groups \mathbb{G}_p are ALM for $p \notin \{1, -1\}$.

We claim that the group \mathbb{G}^1 is also ALM. We have that $\mathbb{G}^1 = \bigcup_{\delta_1, \dots, \delta_n} W_{\delta_1, \dots, \delta_n}$ for $\delta_j \in \{0, 1\}$ where $W_{\delta_1, \dots, \delta_n}$ is the σ -variety defined by the functional equation

$$(\sigma(x_1) + (-1)^{\delta_1} x_1^2, \dots, \sigma(x_n) + (-1)^{\delta_n} x_n^2) = \mathbf{0}$$

Assuming $W_{\delta_1, \dots, \delta_n} \neq \emptyset$, we can find $\bar{w} \in W_{\delta_1, \dots, \delta_n} \neq \emptyset$, then the map $\theta : W_{\delta_1, \dots, \delta_n} \rightarrow \mathbb{G}_2$ given by $\theta(x_1, \dots, x_n) = (w_1 x_1, \dots, w_n x_n)$ is easily checked to be a definable bijection. It follows easily that the property of ALM is inherited by each $W_{\delta_1, \dots, \delta_n}$, hence by \mathbb{G}^1 .

We now claim the following for the situations a. and b.;

$\bigcup_{1 \leq j \leq N} \bar{w}_j T_j \subseteq \overline{V \cap \mathbb{G}^1} = \bigcup_{1 \leq j \leq M} \bar{a}_j T_j$, where the left hand side consists of the union of all maximal *torsion* cosets on V and the right hand side consists of a union of cosets of algebraic subgroups of $\mathbb{G}_m^n(K)$.

By the property of ALM, we clearly have the right hand equality. For the left hand inclusion, suppose that $\bar{w}_j T_j$ defines a maximal torsion coset on V , where \bar{w}_j is chosen in the form given above. Then, by construction, $\bar{w}_j \in \mathbb{G}^1$. Now consider the variety $W \subset \mathbb{G}_m^r(K) \times \mathbb{G}_m^r(K)$ defined by the equations $\langle y_1 - x_1^2, \dots, y_r - x_r^2 \rangle$. This has dimension r and projects dominantly onto the factors $\mathbb{G}_m^r(K)$. We claim that there exists $(x_1, \dots, x_r) \in \mathbb{G}_m^r(K)$, generic over $\text{acl}(\mathcal{Q})$, with $(\bar{x}, \sigma(\bar{x})) \in W$. By compactness, it is sufficient to prove that for any proper closed subvariety Y of $\mathbb{G}_m^r(K)$, defined over $\text{acl}(\mathcal{Q})$, we can find \bar{x} with $(\bar{x}, \sigma(\bar{x})) \in W \setminus (Y \times \mathbb{G}_m^r(K)) \cap W$. This can be done exactly using the axiom scheme for *ACFA*. Now let Φ define the isomorphism of T_j with $\mathbb{G}_m^r(K)$, then $\bar{w}_j \Phi(\bar{x})$ belongs to $\bar{w}_j T_j$ and is generic over the field defining the union of the maximal torsion cosets. By construction, the point $\bar{w}_j \Phi(\bar{x})$ lies inside $V \cap \mathbb{G}^1$. Hence, $\bar{w}_j T_j \subseteq \overline{V \cap \mathbb{G}^1}$ as required.

The functional equation (c) does not define an ALM σ -variety, hence this approach fails (can it be defined using exponents ≥ 1 ??)

Instead, we can use the method given in [2] to handle this case;

Fix prime numbers p and q with $p \neq q$. We claim there exists $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with $\sigma(\omega) = \omega^p$ for all primitive K 'th roots of unity with $(K, p) = 1$ and $\sigma(\omega) = \omega^q$ for all primitive K 'th roots with $K = p^s$. Let \mathbb{Q}_p be the completion of \mathbb{Q} at the prime p and $\mathbb{Q}_p^{unr} = \bigcup_{(K,p)=1} \mathbb{Q}(\xi_K)$ with ξ_K a primitive K 'th root of unity. The residue field of $\mathbb{Q}_p(\xi_K)$ is $F_p(\xi_K)$. The extension is unramified, so $\text{Gal}(\mathbb{Q}_p(\xi_K)/\mathbb{Q}) = \text{Gal}(F_p(\xi_K)/F_p)$, with canonical generator ϕ_p lifting Frobenius given by $\phi_p(\xi_K) = \xi_K^p$. Clearly ϕ_p lifts to an element of $\text{Gal}(\mathbb{Q}_p^{unr}/\mathbb{Q}_p)$ such that $\phi_p(\xi_K) = \xi_K^p$ for all primitive K 'th roots with $(K, p) = 1$. Similarly, we can find ϕ_q with $\phi_q(\xi_K) = \xi_K^q$ for all primitive K 'th roots with $(K, q) = 1$. As $p \neq q$, we have $\phi_q(\xi_K) = \xi_K^q$ for $K = p^s$. By restriction, ϕ_p and ϕ_q define automorphisms of L and M where L is the extension of \mathbb{Q} obtained by adding primitive roots of unity ξ_K with $(K, p) = 1$ and M is obtained by adding primitive roots of unity with $K = p^s$. These fields are linearly disjoint over \mathbb{Q} , hence we can find a single automorphism σ on $\overline{\mathbb{Q}}$ with $\sigma|_K = \phi_p$ and $\sigma|_L = \phi_q$. As usual, we can extend σ to a generic automorphism of K . Now we consider the endomorphism on $\mathbb{G}_m^n(K)$ defined by $p(\sigma)$ where $p(X)$ is the polynomial $(X - p)(X - q)$. We claim that $\mathbb{G}_m^n(K^{cycl}) \subset \text{Ker}(p(\sigma))$. First note that if \bar{w} is a cyclotomic point of order K with $(K, p) = 1$, then we can write $\bar{w} = (\omega_1, \dots, \omega_n)$ with the ω_i primitive roots of unity of

order prime to p . By construction $\sigma(\bar{w}) = p(\bar{w})$, hence $(\sigma - p).(\bar{w}) = \mathbf{1}$. By the same argument, if \bar{w} is a cyclotomic point of order p^s , then $(\sigma - q).(\bar{w}) = \mathbf{1}$. Now suppose that \bar{w} is an arbitrary cyclotomic point of order L . Let $L = p^s K$ where $(K, p) = 1$. We can find integers a and b with $aK + bp^s = 1$, hence $\bar{w} = (\bar{w})^{aK}(\bar{w})^{bp^s} = (\bar{w}_1)^a(\bar{w}_2)^b$ where \bar{w}_1 is cyclotomic of order p^s and \bar{w}_2 is cyclotomic of order K . We then have that;

$$p(\sigma)(\bar{w}) = (p(\sigma)(\bar{w}_1))^a(p(\sigma)(\bar{w}_2))^b = ((\sigma - p).\mathbf{1})^a((\sigma - q).\mathbf{1})^b = \mathbf{1}$$

as required. We now have the following explicit functional equation for $Ker(p(\sigma))$ using coordinates (x_1, \dots, x_n) on $\mathbb{G}_m^n(K)$;

$$p(\sigma).(x_1, \dots, x_n) = \mathbf{1} \quad (2)$$

$$\iff (\sigma - p). \frac{(\sigma(x_1), \dots, \sigma(x_n))}{(x_1^q, \dots, x_n^q)} = \mathbf{1}$$

$$\iff \frac{(\sigma^2(x_1), \dots, \sigma^2(x_n)).(x_1^{pq}, \dots, x_n^{pq})}{((\sigma(x_1))^q, \dots, ((\sigma(x_n))^q)).((\sigma(x_1))^p, \dots, ((\sigma(x_n))^p))} = \mathbf{1}$$

$$\iff ((\sigma^2 x_1)(x_1^{pq}) - (\sigma x_1)^p(\sigma x_1)^q, \dots, (\sigma^2 x_n)(x_n^{pq}) - (\sigma x_n)^p(\sigma x_n)^q) = \mathbf{0}$$

Let $\mathbb{G}_{p,q} = Ker(p(\sigma))$. Using the theorem above, we know that $\mathbb{G}_{p,q}$ is ALM. Moreover, we again have that;

$$\bigcup_{1 \leq j \leq N} \bar{w}_j T_j \subseteq \overline{V \cap \mathbb{G}_{p,q}} = \bigcup \bar{a}_j T_j. \quad (**)$$

where the left hand union is over all maximal *torsion* cosets and the right hand union consists of cosets of algebraic subgroups of $\mathbb{G}_m^n(K)$. We again have the right hand equality by the ALM property. For the left hand inclusion, note that $\bigcup_{1 \leq j \leq N} \bar{w}_j T_j = \overline{V(K^{cycl})}$ and $V(K^{cycl}) \subset V \cap \mathbb{G}_{p,q}$ by construction of $\mathbb{G}_{p,q}$.

We now use the functional equations to obtain explicit bounds on the number of maximal torsion cosets on V . This is done by finding a bound N for the number of irreducible components of $\overline{V \cap G}$ where G is one of the groups $\mathbb{G}_{p,q}, \mathbb{G}^1$. By $(**)$ and Lemma 0.1, this will give a bound N for the number of maximal torsion cosets on V . We require the following lemma (Proposition 2.2.1 in [2])

Lemma 0.3. *Let $P_n(K)$ be n -dimensional projective space, and (K, σ) a difference closed difference field. Let S be a subvariety of $P_n^l(K)$ defined over K . Let*

$$Z = \text{Zariski closure of } \{x \in P_n(K) : (x, \sigma(x), \dots, \sigma^{l-1}(x)) \in S\}.$$

Then $\deg(Z) \leq \deg(S)^{2^{\dim(S)}}$. In particular Z has at most $\deg(S)^{2^{\dim(S)}}$ irreducible components.

Here, $\deg(S)$ is the sum of the multi-degrees of S . It is a straightforward exercise to rephrase this result replacing $P_n(K)$ by $\mathbb{G}_m^n(K)$ and $P_n^l(K)$ by $\mathbb{G}_m^{nl}(K)$.

Now, in the case of the functional equation **(1)** for the cases (a, b) and the functional equation **(2)** which covers cases (a, b, c) , we construct the following varieties W_1 and W_2 .

For coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ on $\mathbb{G}_m^{2n}(K)$;

$$W_1 = \langle (y_1 - x_1^2)(y_1 + x_1^2), \dots, (y_n - x_n^2)(y_n + x_n^2) \rangle$$

For coordinates $(x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_n)$ on $\mathbb{G}_m^{3n}(K)$;

$$W_2 = \langle z_1 x_1^{pq} - y_1^{p+q}, \dots, z_n x_n^{pq} - y_n^{p+q} \rangle$$

A straightforward calculation gives that $\deg(W_1) = 3^n$ and $\dim(W_1) = n$ whereas $\deg(W_2) = (pq + 1)^n$ and $\dim(W_2) = 2n$

As σ fixes V , we have in cases (a, b) that;

$$\bigcup_{1 \leq j \leq N} \bar{w}_j T_j \subseteq \overline{\{x \in \mathbb{G}_m^n(K) : (x, \sigma x) \in V^2 \cap W_1\}}$$

and in cases (a, b, c) that;

$$\bigcup_{1 \leq j \leq N} \bar{w}_j T_j \subseteq \overline{\{x \in \mathbb{G}_m^n(K) : (x, \sigma x, \sigma^2 x) \in V^3 \cap W_2\}}$$

We finally need the following version of Bezout's theorem for counting the components of intersections in multi-projective space;

Lemma 0.4. *Bezout's Theorem*

Let V, W be subvarieties of $\mathbb{P}_n^l(K)$. Let Z_1, \dots, Z_t be the irreducible components of $V \cap W$. Then $\sum_{i=1}^t \deg(Z_i) \leq \deg(V)\deg(W)$.

Applying this to the current situation, we have $\deg(V^2 \cap W_1) \leq \deg(V)^2 3^n$ and $\deg(V^3 \cap W_2) \leq \deg(V)^3 (pq + 1)^n$. We also have that $\dim(V^2 \cap W_1) \leq 2\dim(V)$ and $\dim(V^3 \cap W_2) \leq 3\dim(V)$. Now, using Lemma 0.2, we have that;

$$N \leq (\deg(V)^2 3^n)^{2^{2\dim(V)}} = \exp[2^{2\dim(V)} \log(d^2 3^n)] \text{ in cases } (a, b)$$

and

$$N \leq (\deg(V)^3 (pq + 1)^n)^{2^{3\dim(V)}} \text{ in cases } (a, b, c).$$

where $d = \deg(V)$

Taking $p = 2$ and $q = 3$ gives

$$N \leq (\deg(V)^3 7^n)^{2^{3\dim(V)}} = \exp[2^{3\dim(V)} \log(d^3 7^n)] \text{ in cases } (a, b, c).$$

Using the toric version of Bezout's theorem in [1], we can replace $\deg(V)$ by $\text{vol}(N(V))$, where $N(V)$ is the Newton polytope associated to V . This gives a slightly more refined estimate for subvarieties V of $\mathbb{G}_m^n(K)$. A comparison of the estimates from [4] shows that the estimate in this paper is better in certain situations and worse in others, depending on the dimension and degree of V .

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